

BUNDLES OF CHIRAL BLOCKS AND BOUNDARY CONDITIONS IN CFT [■]

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Abstract

Various aspects of spaces of chiral blocks are discussed. In particular, conjectures about the dimensions of irreducible sub-bundles are reviewed and their relation to symmetry breaking conformal boundary conditions is outlined.

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1 Vertex operator algebras and chiral blocks

Chiral blocks – also known as *conformal* blocks – arise in the study of two-dimensional conformal field theory. In physics terminology, they are correlation “functions” of so-called chiral vertex operators ϕ_μ . Thus, roughly, one deals with objects of the form

$$\langle \phi_{\mu_1}(p_1) \phi_{\mu_2}(p_2) \cdots \phi_{\mu_m}(p_m) \rangle_{(\tilde{C})}, \quad (1)$$

where \tilde{C} is a two-dimensional manifold, some kind of ‘operator product’ between the chiral vertex operators ϕ_{μ_i} ‘sitting’ at $p_i \in \tilde{C}$ is understood, and $\langle \cdots \rangle$ stands for the operation of forming the ‘vacuum expectation value’. Yet, chiral blocks are in general neither functions, nor uniquely determined by these data. For a more detailed understanding several concepts are needed, among them in particular the following.

- First, the notion of a *vertex operator algebra* $\mathfrak{A} = (\mathcal{H}_\Omega, \mathbf{Y}, v_\Omega, v_{\text{vir}})$. Here $\mathcal{H}_\Omega = \bigoplus_n \mathcal{H}_\Omega^{(n)}$ is an infinite-dimensional \mathbb{Z} -graded vector space, with finite-dimensional homogeneous subspaces $\mathcal{H}_\Omega^{(n)}$. \mathcal{H}_Ω is endowed with infinitely many products, which are encoded in the *vertex operator map* $\mathbf{Y}: \mathcal{H}_\Omega \rightarrow \text{End}(\mathcal{H}_\Omega) \otimes_{\mathbb{C}} \mathbb{C}[[t, t^{-1}]]$, mapping \mathcal{H}_Ω to the Laurent series in a formal variable t with values in the endomorphisms of \mathcal{H}_Ω . The *vacuum element* $v_\Omega \in \mathcal{H}_\Omega^{(0)}$ and the *Virasoro element* $v_{\text{vir}} \in \mathcal{H}_\Omega^{(2)}$ are distinguished vectors in \mathcal{H}_Ω , satisfying $\mathbf{Y}(v_\Omega) = \text{id}$ and $(\mathbf{Y}(v; t=0))v_\Omega = v$ (\mathbf{Y} is therefore also known as state-field correspondence).

These quantities are subject to a number of further axioms (see e.g. [18, 19, 32, 42]), mostly not to be spelled out here. We only mention the requirement that the endomorphisms L_n defined by the expansion $\mathbf{Y}(v_{\text{vir}}) = \sum_{n \in \mathbb{Z}} L_n t^{-n-2}$ form a basis of the Virasoro algebra \mathfrak{Vir} . (More precisely, they provide a representation of \mathfrak{Vir} in which the central element acts as a constant multiple, called the rank of \mathfrak{A} , of the identity.)

- For many purposes, it is sufficient to regard the vertex operator algebra as the Lie algebra spanned over \mathbb{C} by the Fourier–Laurent modes of suitable vertex operators $\mathbf{Y}(v)$. Besides \mathfrak{Vir} (i.e. the L_n) and its supersymmetric generalizations, examples are so-called \mathcal{W} -algebras and untwisted affine Lie algebras \mathfrak{g} . The term *chiral algebra* is used both for the proper vertex operator algebra \mathfrak{A} and for the Lie algebra $\mathcal{L}(\mathfrak{A})$.

- There is a collection of irreducible \mathfrak{A} -modules \mathcal{H}_μ , with μ in some index set I . This includes $\Omega \in I$, i.e. the vector space \mathcal{H}_Ω underlying \mathfrak{A} ; this is called the *vacuum sector*.

The modules \mathcal{H}_μ are graded *weight* modules, with finite-dimensional weight spaces, where the weights are with respect to the zero mode L_0 of \mathfrak{Vir} and a suitable collection $\{H_0^i\}$ of other mutually commuting modes in $\mathcal{L}(\mathfrak{A})$. Hence there is the notion of *characters*, i.e. generating functions $\chi_\mu(\tau, \vec{z}) = \text{tr}_{\mathcal{H}_\mu} e^{2\pi i \tau L_0} e^{2\pi i \vec{z} \cdot \vec{H}_0}$ for weight multiplicities.

- When every \mathfrak{A} -module is fully reducible and $|I| < \infty$, one speaks of a *rational* vertex operator algebra, respectively rational CFT. Below we restrict to this case. In a rational theory the modules \mathcal{H}_μ constitute the simple objects of a modular tensor category (provided that \mathfrak{A} is ‘maximal’, which corresponds to non-degeneracy of braiding).

We can now describe chiral blocks more properly; they are certain linear forms

$$B_{\vec{\mu}} : \mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2} \otimes \cdots \otimes \mathcal{H}_{\mu_m} \equiv \vec{\mathcal{H}}_{\vec{\mu}} \rightarrow \mathbb{C} \quad (2)$$

on the tensor product $\vec{\mathcal{H}}_{\vec{\mu}}$ of the relevant irreducible modules \mathcal{H}_{μ_i} . Further, to establish the connection with formula (1) one considers a complex curve \tilde{C} with ordered marked points $p_i \in \tilde{C}$, and identifies for each i the formal variable t with a local holomorphic coordinate ζ_i at $p_i \in \tilde{C}$. (For $\tilde{C} = \mathbb{P}^1$ one may take $t \hat{=} \zeta_i = z - z_i$ with z a quasi-global holomorphic coordinate on \mathbb{P}^1 – say the standard global coordinate on \mathbb{C} for $\mathbb{P}^1 \sim \mathbb{C} \cup \{\infty\}$ – such that $z(p_i) = z_i$. For higher genus curves, the prescription becomes more complicated.) Then for each $v \in \mathcal{H}_{\Omega}$ the vertex operator map provides a chiral vertex operator as appearing in (1), according to $Y(v) \hat{=} \phi_{\Omega}(v; p_i)$. For other sectors $\mu \neq \Omega$, the chiral vertex operators $\phi_{\mu}(v; p_i)$ with $v \in \mathcal{H}_{\mu}$ correspond in an analogous manner to intertwining operators between \mathfrak{A} -modules. Then one sets

$$\langle \phi_{\mu_1}(v_1; p_1) \phi_{\mu_2}(v_2; p_2) \cdots \phi_{\mu_m}(v_m; p_m) \rangle_{(\tilde{C})} = B_{\vec{\mu}}(v_1 \otimes v_2 \otimes \cdots \otimes v_m). \quad (3)$$

Hereby the chiral algebra is interpreted as the “local implementation of the symmetries” of the system at the *insertion points* $\vec{p} = (p_1, p_2, \dots, p_m)$. But we also want to study (3) in its dependence on the insertion points \vec{p} and on the moduli $\vec{\tau}$ of \tilde{C} . This necessitates the construction, for each curve \tilde{C} and number m of insertions, of a suitable “*global* implementation of the symmetries”. Such an implementation, to be called a *block algebra*¹ and denoted by $\mathfrak{A}_{\vec{p}, \tilde{C}}$, is a family (varying with the insertion points and moduli) of subalgebras of the m -fold tensor product of $\mathcal{L}(\mathfrak{A})$, and can be thought of [15] as providing a generalized co-product.

The action of $\mathfrak{A}_{\vec{p}, \tilde{C}}$ on $\vec{\mathcal{H}}_{\vec{\mu}}$ allows us to be specific about the linear forms (2). Namely, one defines the chiral blocks to be the space $B_{\vec{\mu}} = ((\vec{\mathcal{H}}_{\vec{\mu}})^*)^{\mathfrak{A}_{\vec{p}, \tilde{C}}}$ of $\mathfrak{A}_{\vec{p}, \tilde{C}}$ -*singlets* in the algebraic dual $(\vec{\mathcal{H}}_{\vec{\mu}})^*$ – or dually, as the space

$$B_{\vec{\mu}}^* = [\vec{\mathcal{H}}_{\vec{\mu}}]_{\mathfrak{A}_{\vec{p}, \tilde{C}}} \quad (4)$$

of *co-invariants* of $\vec{\mathcal{H}}_{\vec{\mu}}$ with respect to $\mathfrak{A}_{\vec{p}, \tilde{C}}$. In physics terminology, this prescription says that $B_{\vec{\mu}}$ is the space of solutions to the *Ward identities* of the system.

2 Chiral blocks in WZW models

By a *WZW model* one means a conformal field theory for which $\mathcal{L}(\mathfrak{A})$ is an untwisted affine Lie algebra \mathfrak{g} (or, more precisely, its semi-direct sum with Vir) and for which the Virasoro representation is supplied by the affine *Sugawara construction*, which says that the Virasoro generators are quadratic expressions in the generators of \mathfrak{g} , with coefficients proportional to the Killing form of the horizontal subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$.

¹ Unfortunately, it is $\mathfrak{A}_{\vec{p}, \tilde{C}}$ that mathematicians sometimes call the ‘chiral algebra’, see e.g. [27].

Various notions of conformal field theory have very concrete WZW realizations:

- One can regard the affine Lie algebra \mathfrak{g} as being obtained via the *loop construction* from a finite-dimensional simple Lie algebra $\bar{\mathfrak{g}}$; the formal variable t of the vertex operator formalism is closely related to the indeterminate of the loop construction. In this presentation \mathfrak{g} has a basis $\{J_n^a\}$ with $n \in \mathbb{Z}$ and $\{J^a\}$ a basis of $\bar{\mathfrak{g}}$ (together with a central element K and a derivation D), and the simple Lie algebra $\bar{\mathfrak{g}}$ can be identified with the horizontal subalgebra \mathfrak{g}_0 , which is spanned by the zero modes J_0^a .
- The spaces \mathcal{H}_μ are irreducible highest weight modules over \mathfrak{g} with integrable highest weight μ of fixed level $k \in \mathbb{Z}_{>0}$. There are only finitely many such weights μ (in fact the theory is rational), namely those whose horizontal part $\bar{\mu}$ is a dominant integral $\bar{\mathfrak{g}}$ -weight with inner product $(\bar{\mu}, \bar{\theta}^\vee) \leq k$ with the highest coroot $\bar{\theta}^\vee$. For instance, for $\bar{\mathfrak{g}} = \mathfrak{sl}(2)$ only the $\mathfrak{sl}(2)$ -weights $\bar{\mu} = 0, 1, \dots, k$ are allowed.

The vacuum sector is the *basic* \mathfrak{g} -module, which has highest weight $\Omega \equiv k\Lambda_{(0)}$.

- One has $L_0 = -D$, and \vec{H}_0 form a basis of the Cartan subalgebra of $\bar{\mathfrak{g}}$. The characters χ_μ are obtained from the corresponding formal \mathfrak{g} -characters – i.e. elements of the group algebra spanned by formal exponentials in the weights – by interpreting the formal exponentials as functions on weight space. They are convergent for $\Im(\tau) > 0$.
- The block algebra $\mathfrak{A}_{\vec{p}, \tilde{C}}$ is then (for details see e.g. [4, 48, 43]) the tensor product

$$\mathfrak{g}_{\vec{p}, \tilde{C}} = \bar{\mathfrak{g}} \otimes \mathcal{F}_{\vec{p}, \tilde{C}}, \quad (5)$$

with $\mathcal{F}_{\vec{p}, \tilde{C}}$ the algebra of functions holomorphic on $\tilde{C} \setminus \{\vec{p}\}$ and with (at most) finite order poles at the p_i . The action of $\mathfrak{g}_{\vec{p}, \tilde{C}}$ on $\vec{\mathcal{H}}_{\vec{\mu}}$ is given by $(R_{\vec{\mu}}(\bar{x} \otimes f))(v_1 \otimes v_2 \otimes \dots \otimes v_m) := \sum_{i=1}^m v_1 \otimes v_2 \otimes \dots \otimes R_{\mu_i}(x_i) v_i \otimes \dots \otimes v_m$ for $v_1 \otimes v_2 \otimes \dots \otimes v_m \in \vec{\mathcal{H}}_{\vec{\mu}}$, where the $x_i \equiv \bar{x} \otimes f_{p_i}$ are to be regarded as elements of \mathfrak{g} (f_{p_i} denotes the local expansion of f at p_i).

For general CFTs, much less is known about block algebras and their action on tensor products. Roughly, one must ‘couple’ Virasoro-(quasi)primary fields to meromorphic sections of suitable powers of the canonical bundle of \tilde{C} ; in the WZW case this power is zero, hence one deals with functions (5) and can be very explicit. Thus for general CFTs many facets of what is reported below are not at all rigorous – to us it is a major challenge in CFT to improve this.² In contrast, for WZW models already enough is known so as to make precise statements and establish rigorous proofs.

² It is as yet unclear whether the vertex operator framework is broad enough for a rigorous discussion of all issues of interest in CFT, or whether one must resort to formulations involving e.g. von Neumann algebras. (The latter would be unfortunate, as one would give up on treating non-unitary models, like ghost systems in string theory, at an equal footing as unitary ones.) In the WZW case, such a formulation follows by studying the loop group LG of the compact, connected and simply connected real Lie group G whose Lie algebra is the compact real form of $\bar{\mathfrak{g}}$, as well as the associated local loop groups and their representations on Hilbert spaces; compare e.g. [50].

In this context, note that in the vertex operator setting no topology is chosen on the vector spaces \mathcal{H}_λ , i.e. even in the unitary case they are only *pre*-Hilbert spaces. In fact, for certain purposes – e.g. when trying to achieve

3 Bundles of chiral blocks

Chiral block spaces have been studied in quite some detail for several reasons. (In CFT their significance emanates from the fact that they contain the physical correlation functions as special elements, see below.) Among the pertinent results are:

- In all known cases (not only for rational CFTs), for fixed insertion points \vec{p} and fixed moduli $\vec{\tau}$ of \tilde{C} , the space $B_{\vec{\mu}}$ is a *finite-dimensional* vector space $B_{\vec{\mu}}(\vec{p}, \vec{\tau})$. This has a counterpart in the associated tensor category: all morphism spaces in a C^* -tensor category with conjugates and irreducible unit are finite-dimensional [35].
- The spaces $B_{\vec{\mu}}(\vec{p}, \vec{\tau})$ fit together to the total space of a finite rank vector bundle $\mathcal{B}_{\vec{\mu}}$ over the moduli space of genus g complex curves with m ordered marked points.
- Using $\check{\text{Vir}}$ one constructs a projectively flat ‘Knizhnik–Zamolodchikov’ connection on $\mathcal{B}_{\vec{\mu}}$. (Some authors reserve the term ‘chiral block’ for flat sections of $\mathcal{B}_{\vec{\mu}}$.)
- The block bundles are in general not irreducible (i.e. the fibers decompose into a direct sum of vector spaces, in a manner compatible with the transition functions).
- One of the major reasons for independent mathematical interest in chiral blocks is the role played in algebraic geometry by the WZW one-point blocks with vacuum insertion ϕ_{Ω} . Namely, the Picard group of the moduli space $\mathcal{M}_{G, \tilde{C}}$ of holomorphic principal G_c -bundles (with G_c the complexification of G) over \tilde{C} modulo stable equivalence,³ is generated by the determinant line bundle L . $L = \mathcal{O}(\theta)$ is a locally free rank-one sheaf of meromorphic functions on $\mathcal{M}_{G, \tilde{C}}$, where θ is the Theta divisor. Now for every $k \in \mathbb{Z}_{>0}$, the space of holomorphic sections of $L^{\otimes k}$ is canonically isomorphic to the space of one-point blocks on \tilde{C} with insertion ϕ_{Ω} at level k :

$$H^0(\mathcal{M}_{G, \tilde{C}}, L^{\otimes k}) \cong B_{k\Lambda_{(0)}}(\tilde{C}). \quad (6)$$

With traditional methods this ‘space of generalized Theta functions’ had been accessible only in a few special cases (for more information, see e.g. [4, 44]).

4 Dimensions

When studying chiral blocks, the first quantity of interest that comes to mind is the rank of the bundle $\mathcal{B}_{\vec{\mu}}$, i.e. the dimension $N_{\vec{\mu}; \tilde{C}} = \dim B_{\vec{\mu}; \tilde{C}}$ of the spaces $B_{\vec{\mu}; \tilde{C}}$. In CFT, the integers that the generators of $\mathcal{L}(\mathfrak{A})$ act continuously on the (dual) blocks – other topologies than the Hilbert space topology based on the standard norm can be more convenient.

³ $\mathcal{M}_{G, \tilde{C}}$ possesses several other interpretations as well, such as: the set of equivalence classes of flat principal G -bundles; the space of semi-stable holomorphic vector bundles E over \tilde{C} such that the sheaf of sections of the determinant bundle is the structure sheaf of \tilde{C} , $\det E = \mathcal{O}_{\tilde{C}}$; and the phase space $\mathcal{A}_o/\mathcal{G}$ (flat connections modulo gauge transformations) of Chern–Simons gauge theory.

In the first place, these are just bijections of sets. But each of the sets comes equipped with its own natural structures. One can translate those, so that indeed one gets a set with various different interesting structures. A crucial input for establishing these relations is Borel–Weil–Bott theory.

$N_{\lambda,\mu}^\nu \equiv N_{\lambda,\mu,\nu^+;\mathbb{P}^1}$ give the *fusion rules*, i.e. the number $\#(\phi_\lambda \star \phi_\mu \rightsquigarrow \phi_\nu)$ of ‘independent couplings’ between families of fields. Factorization (see section 7 below) implies that the fusion rules constitute the structure constants of a commutative semi-simple associative algebra with unit and involution, which is called the *fusion rule algebra*. They can be expressed in terms of a unitary symmetric matrix S by the *Verlinde formula*

$$N_{\lambda,\mu}^\nu = \sum_{\kappa \in I} S_{\kappa,\lambda} S_{\kappa,\mu} S_{\kappa,\nu}^* / S_{\kappa,\Omega}. \quad (7)$$

It is worth pointing out that the existence of a diagonalizing matrix S obeying (7) is an immediate by-product of the representation theory of fusion rule algebras. The contents of the *Verlinde conjecture* is not formula (7) in itself, but rather that it is one and the same matrix S that appears in (7) and that affords the modular transformation $\tau \mapsto -1/\tau$ on the characters χ_μ . This implies in particular concrete expressions for S , e.g. the Kac–Peterson formula for WZW models, and similarly for coset models and WZW orbifolds.

By factorization (see below), the Verlinde formula (7) generalizes as

$$N_{\vec{\mu};\tilde{C}} = \sum_{\kappa \in I} |S_{\kappa,\Omega}|^{2-2g} \prod_{i=1}^m \frac{S_{\kappa,\mu_i}}{S_{\kappa,\Omega}} \quad (8)$$

to an arbitrary number m of insertions and arbitrary genus g . (8) has been proven rigorously only for WZW models (in particular by algebraic geometry means, cf. e.g. [4, 11] and also [47, 17]). But there is enormous evidence that it holds in general; in particular it was verified for very many theories that the numbers (8) are in $\mathbb{Z}_{\geq 0}$.

5 Traces

The dimensions (8) are only the most basic characteristics of blocks. Other quantities are, of course, of interest as well. As the blocks are in general not irreducible as vector bundles, a natural generalization are the dimensions of irreducible sub-bundles. For many chiral blocks, a non-trivial sub-bundle structure follows from the presence of some group \mathcal{S} of automorphisms σ of $\mathfrak{A}_{\vec{\mu},\tilde{C}}$, which in turn come from automorphisms of \mathfrak{A} . There are then linear bijections Θ_σ between the \mathcal{H}_μ satisfying the twisted intertwiner property $\Theta_\sigma^{-1} \circ Y(\sigma v; z) \circ \Theta_\sigma = Y(v; z)$ and descending to linear maps $\Theta_{\vec{\sigma}}$ on the blocks. The $\Theta_{\vec{\sigma}}$ realize \mathcal{S} projectively, and the sub-bundles are obtained by the simultaneous eigenspace decomposition of the blocks with respect to these maps [24].

Some information on the dimensions of such sub-bundles is available, too. The dimensions are most favorably expressed in terms of traces of the twisted intertwiners $\Theta_{\vec{\sigma}}$, to which they are related by Fourier transformation with respect to the subgroup of \mathcal{S} that corresponds to

the center of a twisted group algebra, where the twist is by the cocycle defining the projectivity of the action on the blocks. Concretely, there are generalizations of the Verlinde conjecture for two important types of automorphisms:

■ First, for automorphisms associated to *simple currents* ϕ_J . A simple current is a unit of the fusion algebra; it can be characterized by the equality $S_{J,\Omega} = S_{\Omega,\Omega}$. Simple currents of WZW models correspond to symmetries of the Dynkin diagram of the underlying affine Lie algebra \mathfrak{g} and thereby to certain outer automorphisms σ_J of \mathfrak{g} . The proposed formula for the traces of the corresponding twisted intertwiners $\theta_J \equiv \theta_{\sigma_J}$ reads [24]

$$\text{Tr}_{\mathcal{B}_{\vec{\mu};\mathbb{P}^1}}(\Theta_{J_1, J_2, \dots, J_m}) = \sum_{\kappa: J_\ell \star \kappa = \kappa} |S_{\kappa, \Omega}|^2 \prod_{i=1}^m \frac{S_{\kappa, \mu_i}^{J_i}}{S_{\kappa, \Omega}}. \quad (9)$$

Here it is assumed that $J_1 \star J_2 \star \dots \star J_m = \Omega$ as well as $J_i \star \mu_i = \mu_i$ for all $i = 1, 2, \dots, m$ (when formally extended to other cases, the expression (9) just yields zero). Further, S^J is the modular matrix for one-point blocks with insertion ϕ_J on an elliptic curve.

■ Second, for automorphisms σ of \mathfrak{A} that preserve v_{vir} one finds [26]

$$\text{Tr}_{\mathcal{B}_{\vec{\mu};\mathbb{P}^1}}(\Theta_{\sigma, \sigma, \dots, \sigma}) = \sum_{\check{\kappa}} |S_{\check{\kappa}, \Omega}^\circ|^2 \prod_{i=1}^m \frac{S_{\check{\kappa}, \mu_i}^\circ}{S_{\check{\kappa}, \Omega}^\circ}. \quad (10)$$

Here S° is an ingredient of the modular S-matrix for an *orbifold* theory that is formed from the original CFT by quotienting out σ . Note that S° has two distinct types of labels; they correspond to the σ -twisted versus the untwisted sector of the orbifold.

In the WZW case, (9) is closely related to a Verlinde formula for non-simply connected groups, see eq. (19) below, while S° coincides [7] with the ordinary S-matrix for (a pair of) twisted affine Lie algebras (being ‘genuinely twisted’ iff σ is outer).

But both (9) and (10) are conjectured for arbitrary rational CFTs – they originate from structures present in every rational CFT. Even for WZW they are far from being proven rigorously – a proof would in particular imply a proof of the Verlinde formula itself. But there are definite ideas for WZW models and also for some derived theories like coset models. In addition, there is enormous numerical evidence: one obtains non-negative integers for the ranks,⁴ even though they are obtained as complicated sums of arbitrary (well, they all lie in a cyclotomic extension of \mathbb{Q}) complex numbers.

⁴ A surprising empirical observation is that in the case of (9) the traces are actually integral themselves, even when the order of σ_J is larger than 2. This is reminiscent of a description of $\dim B_{\vec{\mu};\mathbb{P}^1}$ as the Euler number of a suitable BGG-like complex, and hence suggests that the traces may possess a homological interpretation as well. (On the other hand, the acyclicity result that implies non-negativity of the dimensions cannot generalize. Similar structures have appeared in [12].)

6 Chiral versus full CFT

The sub-bundle structure described by the result (10) can be understood in the framework of orbifold CFTs. As it turns out, the very same chiral concepts play a role in the study of symmetry breaking boundary conditions. Therefore in the rest of this note we address this conceptually different (and at first sight totally unrelated) topic.

As a first step, let us point out that the whole discussion so far concerns what we like to call *chiral conformal field theory*, that is,⁵ CFT on a compact two-dimensional manifold without boundary that has a *complex* structure, or in short, CFT on a complex curve \tilde{C} . The analytic properties of \tilde{C} enter in particular in the definition of block algebras. In contrast, in most applications in physics,⁶ one must consider CFT on a *real* two-dimensional manifold C ⁷ with *conformal* structure. C may be non-orientable or have a boundary, and it does not come with a natural orientation even when it is orientable. We will refer to CFT on C as *full conformal field theory*.

While in chiral CFT one deals with the chiral algebra \mathfrak{A} , chiral vertex operators, chiral blocks, characters, and fusion rules, the key notions in full CFT are fields, correlation functions, the torus partition function, operator products, and boundary conditions.

Chiral CFT is of interest in its own right. But it also serves as a convenient intermediate step in the analysis of full CFT, since it allows to exploit the power of complex geometry. At the geometrical level, the relation between chiral and full conformal field theory is pretty simple. The surface C possesses an oriented two-sheeted Schottky cover \tilde{C} , branched over the boundary ∂C , from which one recovers C by dividing out a suitable anticonformal involution \mathcal{I} . Here are the simplest examples:

- Orientable, no boundary: $C = S^2$ (sphere) $\implies \tilde{C} = \mathbb{P}^1 \sqcup \mathbb{P}^1$, $\mathcal{I}: (z, \dot{z}) \mapsto (\dot{z}^*, z^*)$.
- Orientable, with boundary: $C = D^2$ (disk) $\implies \tilde{C} = \mathbb{P}^1$, $\mathcal{I}: z \mapsto 1/\bar{z}$.
- Non-orientable: $C = \mathbb{RP}^2$ (projective plane / ‘crosscap’) $\implies \tilde{C} = \mathbb{P}^1$, $\mathcal{I}: z \mapsto -1/\bar{z}$.

To implement the transition from \tilde{C} to C at the field theory level requires more work; e.g. for connected \tilde{C} the block algebras are understood only in the simplest cases. For now suffice it to say that, in a rough sense, in many respects the transition amounts to taking two copies of chiral objects. In particular, each single (bulk) field on C comes from *two* chiral vertex operators on \tilde{C} (physically speaking, one has ‘image charges’). Thus it carries *two* chiral labels; we denote it by $\phi_{\mu, \dot{\mu}}$. In addition one must impose some additional constraints and identifications, to be given below.

⁵ Sometimes the term ‘chiral CFT’ is used in a slightly different fashion.

⁶ Among them are string theory and many condensed matter phenomena. But there do exist applications where it is chiral CFT proper that is relevant. An important example is three-dimensional topological field theory – Chern-Simons theory in the WZW case – and thereby the (fractional) quantum Hall effect. (While a priori in the quantum Hall effect there is thus no natural place for modular invariance on the torus, arguments assigning a physical role to it were given in the literature.)

⁷ You might have already wondered why the symbol \tilde{C} was used above rather than just C .

Before proceeding, let us recall that for WZW models various structures can be made fully explicit which for other classes of CFTs are not yet worked out in detail. Fortunately, this is an issue mainly for chiral CFT. Once the chiral theory is taken for granted, considerations in full CFT turn out to be essentially model independent.

7 Correlation functions

Correlation functions are the ‘vacuum expectation values’ of suitable products of ‘(quantum) fields’. They constitute the quantities of most direct interest in applications. For instance, by integrating them over moduli space one obtains string scattering amplitudes. A message to be remembered is that fields and their correlation functions are objects in *full* CFT and hence ‘live’ on the quotient C of \tilde{C} , while chiral blocks ‘live’ on \tilde{C} . Thus the blocks *cannot* be physical correlation functions; rather, a correlation function for C is a specific element in a corresponding space of blocks on \tilde{C} .⁸ That element is determined by various constraints, coming in three types:

- *Locality*: Correlators are (single-valued) functions of the insertion points p_i – unlike generic sections of the block bundle, which typically is not a trivial vector bundle.
- *Locality'*: They are also functions of the moduli of C (modulo the Weyl anomaly).
- *Factorization*: They are compatible with desingularization. This amounts to a restriction on the allowed intermediate states that contribute in singular limits, and is thereby closely related to the existence of operator product expansions.

Technically: \tilde{C} is a stable algebraic curve with at worst ordinary double points as singularities. When \hat{C} is a partial desingularization of \tilde{C} that resolves a double point $p \in \tilde{C}$ in two points $p', p'' \in \hat{C}$, then factorization gives a canonical isomorphism

$$\bigoplus_{\nu \in I} B_{\vec{\mu}, \nu, \nu^+; \hat{C}} \cong B_{\vec{\mu}, \tilde{C}}. \quad (11)$$

A priori neither existence nor uniqueness of a solution to these constraints is clear.

A prominent example is provided by the correlator for $m=0$ and $g=1$. Then C is a torus, and \tilde{C} is the disconnected sum $E_\tau \sqcup E_{\bar{\tau}}$ of two elliptic curves with opposite orientation, $\bar{\tau} = -\tau^*$. The 0-point correlator on C is the torus partition function Z , while a basis for the 0-point blocks on \tilde{C} are tensor products of (Vir-specialized) irreducible characters χ_μ and $\dot{\chi}_{\dot{\mu}}$. So Z is a sesqui-linear combination of characters:

$$Z(\tau) = \sum_{\mu, \dot{\mu}} Z_{\mu, \dot{\mu}} \chi_\mu(\tau) (\dot{\chi}_{\dot{\mu}}(\tau))^*. \quad (12)$$

Z is highly constrained by the property of *modular invariance*, i.e. locality with respect to the modulus τ of E_τ . The solution of this constraint is of interest in its own right.

⁸ Or what is the same, after choosing some (natural) basis in the block space: a specific linear combination of basis blocks. When \tilde{C} is disconnected, this is usually written as a sesqui-linear combination of separate basis blocks for the two connected components of \tilde{C} .

8 Boundary conditions

Solving the factorization and locality constraints is not easy at all, in general, and little is known about uniqueness.⁹ But for special correlation functions, which are still of great interest, a lot can be done explicitly and in much generality. An especially fortunate example is given by the 1-point functions $\langle \phi_{\mu,\dot{\mu}} \rangle$ for bulk fields on the disk. These are important because, due to factorization, there is only a small number of basic building blocks. As long as only closed orientable surfaces C are studied, already the 3-point functions on S^2 are sufficient. In the general situation, in addition the 1-point functions $\langle \phi_{\mu,\dot{\mu}} \rangle$ on the disk and on \mathbb{RP}^2 are needed (as well as 3-point functions for boundary fields, which correspond to open string vertex operators) [34, 16, 39, 40].

The chiral blocks for $\langle \phi_{\mu,\dot{\mu}} \rangle$ are two-point blocks on the Schottky cover \mathbb{P}^1 of the disk. Now one has $N_{\mu,\dot{\mu}} = \delta_{\dot{\mu},\mu^+}$, so only a single coefficient needs to be determined:

$$\langle \phi_{\mu,\dot{\mu}}(v \otimes v') \rangle_a = \mathcal{R}_{\mu,\dot{\mu};\Omega}^a B_{\mu,\dot{\mu}}(v \otimes v'). \quad (13)$$

Again by factorization, the complex number $\mathcal{R}_{\mu,\dot{\mu};\Omega}^a$ can be interpreted as a *reflection coefficient*, which appears in the bulk-boundary operator product

$$\phi_{\mu,\dot{\mu}}(z) \sim \sum_{\nu \in I} (1 - |z|^2)^{-2\Delta_\mu + \Delta_\nu} \mathcal{R}_{\mu,\dot{\mu};\nu}^a \Psi_\nu^{a,a}(\arg z) \quad \text{for } |z| \rightarrow 1. \quad (14)$$

For closed orientable C it is generally expected that the constraints possess a unique solution. In contrast, the one-point functions on the disk are in general not unique, but an additional label a is needed. This indicates that the disk can come with several distinct *boundary conditions* labelled by a . A boundary condition is essentially the same as a consistent collection of one-point functions of bulk fields on the disk.

One of the most fundamental tasks in CFT is to determine, assuming the theory to be known at the chiral level, all consistent conformally invariant boundary conditions.

9 Classifying algebras

Thus let us address the task of determining the conformal boundary conditions for a CFT that is known at the chiral level. Unfortunately the requirement of conformal invariance is rather weak, simply because \mathfrak{A} is typically much larger than just \mathfrak{Vir} . As a result, there will in general (e.g. already for free boson theories) be *infinitely many* conformal boundary conditions. Usually they will be difficult to survey.

A pragmatic way out of this dilemma is to impose invariance under all of \mathfrak{A} , or at least under a sufficiently large consistent chiral subalgebra $\bar{\mathfrak{A}}$ of \mathfrak{A} , rather than only under the Virasoro

⁹ Just think of the case of the torus partition function (12), where the constraint (modular invariance) looks quite innocent, but is still hard to solve. See e.g. [28] for review and references.

algebra $\mathcal{V}ir$. (Here ‘invariance’ means that the behavior at the boundary ∂C , cf. formula (14), is identical for all bulk fields that are associated to vectors in a given $\bar{\mathfrak{A}}$ -submodule of an \mathfrak{A} -module \mathcal{H}_μ .) Then one can achieve a ‘rational’ situation, with only finitely many boundary conditions. Now by comparing two different factorization limits of the two-point function $\langle \phi_{\mu,\dot{\mu}} \phi_{\nu,\dot{\nu}} \rangle_a$, one can show that

$$\mathcal{R}_{\lambda,\dot{\lambda};\Omega}^a \mathcal{R}_{\mu,\dot{\mu};\Omega}^a = \sum_{\nu} \tilde{N}_{\lambda,\mu}^{\nu} \mathcal{R}_{\nu,\dot{\nu};\Omega}^a \quad (15)$$

with numbers $\tilde{N}_{\lambda,\mu}^{\nu}$ which are combinations of fusing matrices and operator product coefficients. At first glance, these expressions look very complicated. But there is a crucial insight: manifestly, $\tilde{N}_{\lambda,\mu}^{\nu}$ does not depend on the boundary condition a .

This observation allows us to interpret the reflection coefficients $\mathcal{R}_{\mu,\dot{\mu};\Omega}^a$ as furnishing a one-dimensional irreducible representation of an algebra $\mathcal{C}(\bar{\mathfrak{A}})$ with structure constants $\tilde{N}_{\lambda,\mu}^{\nu}$, termed [23] the *classifying algebra*. The results of [10] may be summarized by the statement that the classifying algebra $\mathcal{C}(\mathfrak{A})$ for boundary conditions preserving the full bulk symmetry \mathfrak{A} (and with charge conjugation as torus partition function) is nothing but the fusion algebra of the CFT. Thus $\mathcal{C}(\mathfrak{A})$ is a semi-simple associative algebra, its structure constants are expressible through the Verlinde matrix S as in (7), and both a basis of $\mathcal{C}(\mathfrak{A})$ and the boundary conditions a are labelled by the set I of chiral labels μ . (Yet, an explicit verification of $\tilde{N}_{\lambda,\mu}^{\nu} = N_{\lambda,\mu}^{\nu}$ was achieved [39] only in special cases where the relevant operator products and fusing matrices are known.)

When $\bar{\mathfrak{A}} \neq \mathfrak{A}$, then the situation is more complicated, though the factorization arguments go through. For such *symmetry breaking boundary conditions* one finds:

- One still has one-dimensional irreducible representations of some algebra $\mathcal{C} = \mathcal{C}(\bar{\mathfrak{A}})$.
- But the correlation functions are different. Namely, they are formed as different combinations of the chiral blocks for \mathfrak{A} -descendant fields that are $\bar{\mathfrak{A}}$ -primaries.
- The labelling $\{\tilde{\mu}\}$ of basis elements of \mathcal{C} and $\{a\}$ of boundary conditions is more subtle. In particular the two sets of labels are distinct; both differ from the set I .
- When the unbroken part of the bulk symmetries constitutes the fixed point algebra

$$\bar{\mathfrak{A}} = \mathfrak{A}^G \quad (16)$$

with respect to any finite abelian group G of automorphisms of \mathfrak{A} , then the boundary conditions can be analyzed via G -orbifold and simple current techniques.

10 Interlude: Simple current extensions

One of the CFT concepts that was instrumental for arriving at conjectures (9) and (10) is the *simple current extension* of a rational CFT. It will show up again in the study of boundary conditions below. Assume that the following data are given:¹⁰

- A set $\{\chi_\mu\}$ ($\mu \in I$, $|I| < \infty$) of functions of $\tau \in \mathbb{C}$, convergent for $\Im(\tau) > 0$ and forming a basis of a unitary module V over $\text{SL}(2, \mathbb{Z})$ for which $S = S^t$ and $T = \text{diag}$.
- A vacuum label $\Omega \in I$, satisfying $S_{\Omega, \mu} \in \mathbb{R}_{>0}$ for all $\mu \in I$, and an involution $\mu \mapsto \mu^+$ on I such that $\Omega^+ = \Omega$ as well as $S_{\lambda, \mu^+} = S_{\lambda, \mu}^*$ and $T_{\mu^+} = T_\mu$ for all $\lambda, \mu \in I$.
- A subset $\mathcal{G} \subseteq I$ such that $S_{J, \Omega} = S_{\Omega, \Omega}$ and $T_J = T_\Omega$ for all $J \in \mathcal{G}$.
(In CFT terms: $J \in \mathcal{G}$ has the same quantum dimension (namely unity) and the same conformal weight mod \mathbb{Z} (namely zero) as Ω , i.e. is an integer spin simple current.)
- The numbers $N_{\lambda, \mu}^\nu$, regarded as *defined* by formula (7), are non-negative integers.

In this situation one defines a *fusion ring* with product ‘ \star ’ on the vector space spanned by $\{\varphi_\mu \mid \mu \in I\}$ by $\varphi_\lambda \star \varphi_\mu := \sum_{\nu \in I} N_{\lambda, \mu, \nu} \varphi_\nu$, and can show rigorously [41]:

- \mathcal{G} is a finite abelian group w.r.t. ‘ \star ’ – the group of units of the fusion ring.
- \mathcal{G} organizes I into orbits $[\mu] := \{J\mu \mid J \in \mathcal{G}\}$, with $\phi_J \star \phi_\mu =: \phi_{J\star\mu} \equiv \phi_{J\mu}$.
- Defining the *stabilizer subgroup* $\mathcal{S}_\lambda := \{J \in \mathcal{G} \mid J\lambda = \lambda\}$, the combination

$$Z(\tau) = \sum_{\substack{[\mu]: \mu \in I, \\ T_{J\mu} = T_\mu \quad \forall J \in \mathcal{G}}} (|\mathcal{S}_\mu| \cdot \left| \sum_{J \in \mathcal{G}/\mathcal{S}_\mu} \chi_{J\mu}(\tau) \right|^2) \quad (17)$$

is $\text{SL}(2, \mathbb{Z})$ -invariant. Z is called a simple current extension modular invariant.¹¹

To justify this name, one must be able to interpret (17) as the diagonal invariant for some extended CFT. This was achieved in [22], where the following was proven:

- The extended labels are equivalence classes of pairs $[\mu, \hat{\psi}]$ with $T_{J\mu} = T_\mu$ and $\hat{\psi}$ a character of the *untwisted stabilizer* $\mathcal{U}_\mu := \{J \in \mathcal{S}_\mu \mid F_\mu(J, J') = 1 \quad \forall J' \in \mathcal{S}_\mu\} \subseteq \mathcal{S}_\mu$. Here F_μ is an alternating bi-homomorphism on \mathcal{S}_μ , and hence the commutator cocycle $F_\mu(J, J') = \mathcal{F}_\mu(J, J') / \mathcal{F}_\mu(J', J)$ for some cohomology class $\mathcal{F}_\mu \in H^2(\mathcal{S}_\mu, \text{U}(1))$ [22, 3, 36, 25]. Thus the group algebra $\mathbb{C}\mathcal{U}_\mu$ is isomorphic to the center of the twisted group algebra $\mathbb{C}_{\mathcal{F}_\mu}\mathcal{S}_\mu$, implying that the inclusion $\mathcal{U}_\mu \subset \mathcal{S}_\mu$ is of square index d_μ^2 , with d_μ the dimension of the irreducible $\mathbb{C}_{\mathcal{F}_\mu}\mathcal{S}_\mu$ -representations.¹²

¹⁰ While usually this is formulated by saying that one has some CFT with corresponding properties (and indeed there *are* many CFTs with those properties), here we need not refer directly to CFT.

¹¹ Many of these modular invariants are interesting. Examples include the D_{even} type invariants of the $\mathfrak{sl}(2)$ WZW model and the invariant $Z = |\chi_1 + \chi_{35} + \chi_{35'} + \chi_{35''}|^2 + 4|\chi_{28}|^2$ for D_4 level 2.

¹² F_μ enters in calculations at various places; that for $F_\mu \neq 1$ everything still nicely fits together [25, 26] is a strong consistency check. Non-trivial F_μ appear naturally via products of simple currents that individually have $T_J = -T_\Omega$, e.g. when one deals with tensor products of subtheories, such as in Gepner type string compactifications.

In the D_4 example of footnote 11, one finds $\mathcal{S}_{28} = \mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$ but $\mathcal{U}_{28} = \{\Omega\}$. Thus for $\mu = \Lambda_{(2)} \hat{=} 28$ there is only a single extended character $\chi_{28}^{\text{ext}} = 2 \cdot \chi_{28}$.

- The summands in (17) are to be read as $|\mathcal{U}_\mu| \cdot |\chi_{[\mu]}^{\text{ext}}|^2$, i.e. for each $[\mu]$ there are $|\mathcal{U}_\mu|$ many extended irreducible characters $\chi_{[\mu, \hat{\psi}]}^{\text{ext}}$. Correspondingly the decompositions

$$\mathcal{H}_{[\mu, \hat{\psi}]}^{\text{ext}} = \bigoplus_{J \in \mathcal{G}/\mathcal{S}_\mu} \mathbb{C}^{d_\mu} \otimes \mathcal{H}_{J\mu} \quad (18)$$

hold, and $\chi_{[\mu, \hat{\psi}]}^{\text{ext}} = d_\mu \cdot \sum_{J \in \mathcal{G}/\mathcal{S}_\mu} \chi_{J\mu}$ is the character of the extended module $\mathcal{H}_{[\mu, \hat{\psi}]}^{\text{ext}}$.

- Given, for every $J \in \mathcal{G}$, a unitary matrix S^J satisfying the $\text{SL}(2, \mathbb{Z})$ relations as well as $S_{\lambda, \mu}^J = S_{\mu, \lambda}^{J^{-1}}$ ($J \in \mathcal{S}_\lambda \cap \mathcal{S}_\mu$) and $S^\Omega = S$, the modular S-transformation matrix S^{ext} of the functions $\chi_{[\mu, \hat{\psi}]}^{\text{ext}}$ is obtained by sandwiching the S^J between group characters:

$$S_{[\lambda, \hat{\psi}_\lambda], [\mu, \hat{\psi}_\mu]}^{\text{ext}} = \frac{|\mathcal{G}|}{[|\mathcal{S}_\lambda| |\mathcal{U}_\lambda| |\mathcal{S}_\mu| |\mathcal{U}_\mu|]^{1/2}} \sum_{J \in \mathcal{U}_\lambda \cap \mathcal{U}_\mu} \hat{\psi}_\lambda(J) S_{\lambda, \mu}^J \hat{\psi}_\mu(J)^*. \quad (19)$$

One has $S_{J', \lambda, \mu}^J = (T_\mu / T_{J'\mu}) F_\mu(J, J') S_{\lambda, \mu}^J$. S^{ext} is proven to be unitary and symmetric and to satisfy the $\text{SL}(2, \mathbb{Z})$ relations, and it was checked in a huge number of examples that it produces non-negative integers when inserted in the Verlinde formula [22].

- There is evidence [3] that S^J is the modular S-matrix for the one-point blocks on the torus with insertion J . For WZW or coset models, S^J is the Kac–Peterson matrix of the *orbit Lie algebra* that is related¹³ to \mathfrak{g} by a folding of the Dynkin diagram.
- Simple currents of WZW models correspond to the elements of the center of the relevant covering group G . It follows that (19) appears in the Verlinde formula for non-simply connected groups. (This result was checked in [5] for some simple cases.)

11 The classifying algebra for finite abelian G

A systematic classification of boundary conditions has been achieved for all cases where $\bar{\mathfrak{A}}$ is given as in (16), with finite abelian automorphism group G [25, 26]. (The simplest case $G = \mathbb{Z}_2$ includes e.g. Dirichlet boundary conditions for free bosons.)

All basic ingredients are already known¹⁴ from chiral CFT. In particular:

- The label sets $\{\tilde{\mu}\}$ for the basis of $\mathcal{C}(\bar{\mathfrak{A}})$ and $\{a\}$ for boundary conditions arise as two different deviations from the labels appearing in (19). For $\tilde{\mu}$, one has a character of \mathcal{S}_μ rather than of \mathcal{U}_μ , and no orbit is to be taken, but still the requirement $T_{J\mu} = T_\mu$ is kept, while a has the same form as extended labels, but now $T_{J\mu} \neq T_\mu$ is allowed:

$$\boxed{\tilde{\mu} = (\bar{\mu}, \psi)} \quad \text{and} \quad \boxed{a = [\bar{\rho}(a), \hat{\psi}]} \quad \text{with} \quad \psi \in \mathcal{S}_\mu^*, \hat{\psi} \in \mathcal{U}_{\bar{\rho}(a)}^* \quad (20)$$

¹³ See [21, 20], and also [37, 3, 45, 9, 8, 49, 38, 33] for background material and related work.

¹⁴ In particular one can make use of the fact that simple current extension by $\mathcal{G} \cong G^*$ provides the inverse operation to forming the orbifold with respect to the finite abelian group G . This way one can exploit both orbifold techniques and the simple current framework sketched in section 10.

(recall $\mathcal{U}_{\tilde{\lambda}} \subseteq \mathcal{S}_{\tilde{\lambda}} \subseteq \mathcal{G}$). (20) follows by heuristic considerations resembling ideas in [10].

- Comparing with (19), we can make an educated guess for a diagonalizing matrix:

$$\tilde{S}_{(\tilde{\lambda}, \psi_{\tilde{\lambda}}), [\tilde{\rho}, \hat{\psi}_{\tilde{\rho}}]} = \frac{|\mathcal{G}|}{[|\mathcal{S}_{\tilde{\lambda}}| |\mathcal{U}_{\tilde{\lambda}}| |\mathcal{S}_{\tilde{\rho}}| |\mathcal{U}_{\tilde{\rho}}|]^{1/2}} \sum_{J \in \mathcal{S}_{\tilde{\lambda}} \cap \mathcal{U}_{\tilde{\rho}}} \psi_{\tilde{\lambda}}(J) S_{\tilde{\lambda}, \tilde{\rho}}^J \hat{\psi}_{\tilde{\rho}}(J)^*. \quad (21)$$

- Then $\mathcal{C}(\bar{\mathfrak{A}})$ is *defined* by prescribing the Verlinde-like formula featuring \tilde{S} :

$$\tilde{N}_{\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}} := \sum_a \tilde{S}_{\tilde{\lambda}, a} \tilde{S}_{\tilde{\mu}, a} \tilde{S}_{\tilde{\nu}, a} / \tilde{S}_{\tilde{\Omega}, a}. \quad (22)$$

The structure constants are obtained from (22) by raising the third index via $\tilde{N}_{\tilde{\lambda}, \tilde{\mu}, \tilde{\Omega}}$.

There is as yet no rigorous derivation of formula (21). But once (21) and (22) are taken for granted, $\mathcal{C}(\bar{\mathfrak{A}})$ can be studied with full rigor. In particular one shows [25, 26]:

- \tilde{S} is (weighted) unitary. (Note that it is even non-trivial that \tilde{S} is a square matrix.)
- $\mathcal{C}(\bar{\mathfrak{A}})$ is a semi-simple commutative associative algebra with unit element $\tilde{\Omega} = \tilde{\Omega}$ (the vacuum sector of the G -orbifold).
- The structure constants $\tilde{N}_{\tilde{\lambda}, \tilde{\mu}}^{\tilde{\nu}}$ of $\mathcal{C}(\bar{\mathfrak{A}})$ are diagonalized by the matrix (21). The irreducible $\mathcal{C}(\bar{\mathfrak{A}})$ -representations R_a are one-dimensional and labelled by the boundary labels a ; they yield the reflection coefficients as $\mathcal{R}_{\tilde{\mu}, \tilde{\mu}^+; \tilde{\Omega}}^a = R_a(\phi_{\tilde{\mu}}) = \tilde{S}_{\tilde{\mu}, a} / \tilde{S}_{\tilde{\Omega}, a}$.
- As an algebra over \mathbb{C} , $\mathcal{C}(\bar{\mathfrak{A}})$ decomposes into ideals as

$$\mathcal{C}(\bar{\mathfrak{A}}) \cong \bigoplus_{g \in G} \mathcal{C}^{(g)}(\bar{\mathfrak{A}}). \quad (23)$$

The ideal $\mathcal{C}^{(g)}(\bar{\mathfrak{A}})$ plays the role of a classifying algebra for boundary conditions of definite *automorphism type* g . The corresponding boundary states are linear combinations of g -twisted chiral blocks, which obey g -twisted Ward identities.¹⁵

- The ideal $\mathcal{C}^{(e)}(\bar{\mathfrak{A}})$ appearing in (23) is precisely the fusion rule algebra of \mathfrak{A} .
- It is plausible that orbifolding can be understood in terms of the folding of fusion graphs, and that the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$ thus coincides with the corresponding Pasquier algebra [6].¹⁶
- The statements above refer to the charge conjugation torus partition function. More general results follow via *T-duality* symmetries, which are similar to those of free boson theories, acting compatibly on the boundary conditions and on the torus partition function.

¹⁵ For consistent subalgebras that are not fixed point algebras, there exist boundary conditions which do not possess an automorphism type. Examples of such boundary conditions are e.g. known for the \mathbb{Z}_2 -orbifold of a free boson and for the E_6 -type invariant of the $\mathfrak{sl}(2)$ WZW model.

¹⁶ At least for cyclic G – for non-cyclic G one must be aware of the possibility of having non-trivial two-cocycles \mathcal{F}_{μ} .

Also, in practice this is difficult to check, because the Pasquier algebra is obtained by an algorithm which does not directly produce uniform formulae for all rational CFTs. The identification seems to be established so far only for $G = \mathbb{Z}_2$ in $\mathfrak{sl}(2)$ WZW models and Virasoro minimal models.

Knowing the classifying algebra $\mathcal{C}(\tilde{\mathfrak{A}})$ explicitly, a variety of consistency checks can be made. Most importantly, one can prove [25] integrality of the coefficients of characters in the annulus amplitude (open string partition function). This integrality is often used as the starting point for studying boundaries; here it rather serves as an independent check.

Finally we remark that one can express the structure constants $\tilde{N}_{\lambda, \tilde{\mu}}^{\tilde{\nu}}$ as well as the annulus coefficients through traces of twisted intertwining operators $\Theta_{\tilde{\sigma}}$ on chiral block spaces. This yields the announced connection to the topic studied in section 5.

12 Conclusions and outlook

Let us summarize by telling what we regard as the two main messages:

- First, there exists a close relation between the sub-bundle structure of chiral blocks (leading to the trace formula (10)) and symmetry breaking boundary conditions.
- Second, there is a systematic classification of all boundary conditions leaving unbroken a fixed point algebra $\tilde{\mathfrak{A}} = \mathfrak{A}^G$ with respect to an arbitrary finite abelian group G . Concretely, one has a general prescription, valid for all rational CFTs, for the classifying algebra, with structure constants expressed through known *chiral* data.

Our results illustrate that the space of boundary conditions has a rich and unexpectedly nice structure. We believe that many more issues are accessible quantitatively.

Among possible extensions of the work outlined above we mention:

- One should find the diagonalizing matrix \tilde{S} of $\mathcal{C}(\tilde{\mathfrak{A}})$ when $\tilde{\mathfrak{A}} \neq \mathfrak{A}^G$ for any group G .
- 2-d boundary conditions can be understood in terms of 3-d topological theory [14].
- Non-orientable surfaces, e.g. one-point functions on \mathbb{RP}^2 and the partition functions of the Klein bottle and the Möbius strip, are studied in [39] and [31, 14, 29].
- One should look for a geometric interpretation of boundary conditions for non-flat backgrounds. For WZW models this is indeed available: one obtains ‘fuzzy’ versions of (possibly twisted) conjugacy classes of the group manifold G [2, 1, 30, 13, 46].
- More explicit information on the chiral data for further classes of models is highly welcome.
- Applications to string theory include, e.g., the complete analysis of concrete compactifications and a more systematic understanding of tadpole cancellation.

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